Exercise 5

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

$$y''' - y' = 0,\ y(0) = 2,\ y'(0) = y''(0) = 1$$

Solution

Let

$$y'''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned}\int_0^x y'''(t)\,dt &= \int_0^x u(t)\,dt \\ y''(x) - y''(0) &= \int_0^x u(t)\,dt\end{aligned}$$

Substitute $y''(0) = 1$ and bring it to the right side.

$$y''(x) = 1 + \int_0^x u(t)\,dt \tag{2}$$

Integrate both sides again from 0 to x.

$$\begin{aligned}\int_0^x y''(s)\,ds &= \int_0^x \left[1 + \int_0^s u(t)\,dt\right] ds \\ y'(x) - y'(0) &= x + \int_0^x \int_0^s u(t)\,dt\,ds\end{aligned}$$

Substitute $y'(0) = 1$ and bring it to the right side.

$$y'(x) = 1 + x + \int_0^x \int_0^s u(t)\,dt\,ds$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned}v &= \int_0^s u(t)\,dt \qquad & dw &= ds \\ dv &= u(s)\,ds & w &= s\end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned}y'(x) &= 1 + x + s\int_0^s u(t)\,dt\bigg|_0^x - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= 1 + x + \int_0^x (x-t)u(t)\,dt \qquad (3)\end{aligned}$$

Integrate both sides again from 0 to x.

$$\int_0^x y'(r)\,dr = \int_0^x \left[1 + r + \int_0^r (r-t)u(t)\,dt\right] dr$$

$$y(x) - y(0) = x + \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

Substitute $y(0) = 2$ and bring it to the right side.

$$y(x) = 2 + x + \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

In order to evaluate the double integral, switch the order of integration so that dr comes first.

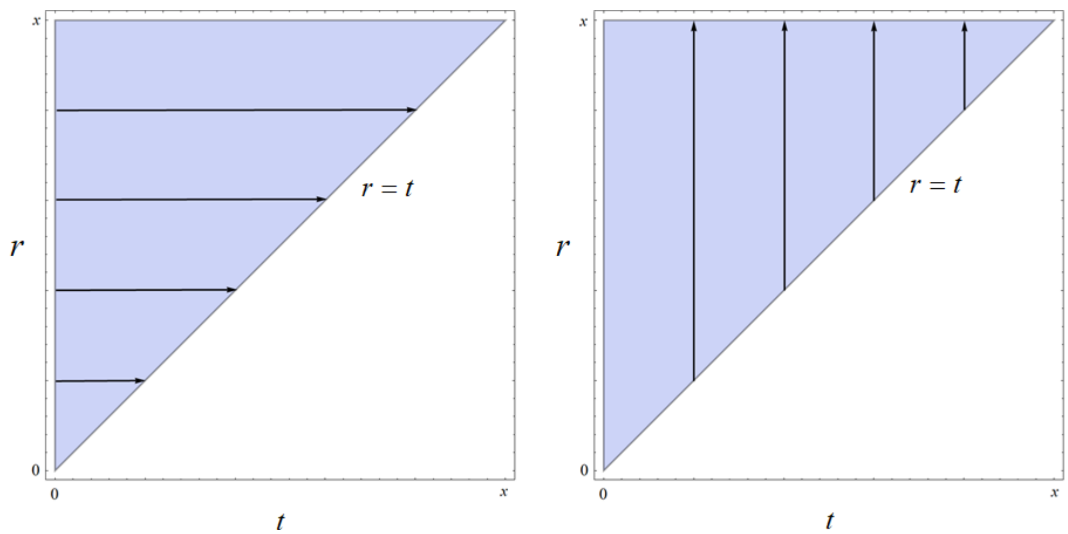

Figure 1: The current mode of integration in the tr-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned} y(x) &= 2 + x + \frac{x^2}{2} + \int_0^x \int_t^x (r-t)u(t)\,dr\,dt \\ &= 2 + x + \frac{x^2}{2} + \int_0^x \left[\frac{(r-t)^2}{2}\right]\Bigg|_t^x u(t)\,dt \\ &= 2 + x + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2} u(t)\,dt \\ &= 2 + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \end{aligned} \tag{4}$$

Substitute equations (1), (2), (3), and (4) into the original ODE.

$$y''' - y' = 0 \quad \rightarrow \quad u(x) - \left[1 + x + \int_0^x (x-t)u(t)\,dt\right] = 0$$

Therefore, the equivalent Volterra integral equation is

$$u(x) = 1 + x + \int_0^x (x-t)u(t)\,dt.$$